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ON COMPLETE SYMMETRIC FUNCTIONS.

By DR. E. D. ROE, JR., Syracuse University.

PART III. PROBLEMS AND TABLES.

1. Suppose we are asked to find the coefficient of x^r in the development $\frac{x+1}{x^2-3x+2}$. As we can find the roots of the denominator, we may proceed as follows:

$$\frac{x+1}{x^2-3x+2}=\frac{A}{1-x}+\frac{B}{2-x}=\frac{2}{1-x}-\frac{3}{2-x}=2(1-x)^{-1}-3(2-x)^{-1},$$

whence the coefficient of x^r is easily found to be $2-3.2^{-r-1}$. But suppose the degree of the denominator is greater than 2; it will then be generally difficult to find the roots, and a general equation of degree greater than four cannot be solved algebraically. We recur to the example stated in (17). We have by using (21),

$$(54)\qquad \phi(x)=\frac{c_1+2c_2x+\text{.........}+(m+1)x^m}{1+b_1x+\text{.........}+b_nx^n}=[c_1+2c_2x+\text{.........}+(m+1)x^m]$$

$$\times(t_0+t_1x+t_2x^2+\text{.........});$$

whence

(55) the coefficient u_{r-1} of $x^{r-1}=c_1t_{r-1}+2c_2t_{r-2}+\text{.........}+rc_rt_0$
or by regarding (21), we see that it can be expressed as

$$(56) \qquad u_{r-1}=(-1)^{r-1}\begin{vmatrix} c_1 & 1 & 0 \ldots\ldots 0 \\ 2c_2 & b_1 & 1 \;\; 0 \ldots 0 \\ \vdots & \vdots & \\ \vdots & \vdots & 1 \\ rc_r & b_{r-1} & \ldots\ldots b_1 \end{vmatrix}.$$

If $c_1=b_1$, $c_2=b_2$, ..., $c_r=b_r$, this becomes $-s_r$ in view of (31), as it should, as one sees by differentiating (29) with respect to x. Formula (26) shows how to express the t's in terms of b's, and hence (56) or (55) taken in connection with (26) gives the general solution of the problem.*

2. To find the value of $\sum\limits_{\kappa=0}^{\kappa=\infty}\dfrac{\kappa^r}{\kappa!}$.

This problem while not directly employing the t's is given as an example which shows how symmetric functions may be used to solve a problem without the use of the individual roots. In this case there is an infinite number of them. Put

$$(57) \qquad ee^x=c_0+c_1x+c_2x^2+\ldots\ldots+c_rx^r+\ldots\ldots$$

where it is easily seen that

$$(58) \qquad c_0=e, \quad c_r=\frac{1}{r!}\sum_{\kappa=0}^{\kappa=\infty}\frac{\kappa^r}{\kappa!};$$

with $b_r=c_r/c_0$, we have from (57)

$$\log c_0+\log(1+b_1x+\ldots\ldots+b_rx^r+\ldots\ldots)=e^x=1+x+\frac{x^2}{2!}+\ldots\ldots+\frac{x^r}{r!}+\ldots\ldots,$$

or by simplifying and using (29),

$$(59) \qquad \log(1+b_1x+b_2x^2+\ldots\ldots)=-\Sigma\frac{s_r\,x^r}{r}=x+\frac{x^2}{2!}+\ldots\ldots+\frac{x^r}{r!}+\ldots\ldots;$$

whence

$$(60) \qquad s_r=-\frac{1}{(r-1)!}.$$

By (37),

$$(61) \qquad b_r=\Sigma(-1)^{\kappa_1+\ldots\ldots+\kappa_r}\frac{1}{\kappa_1!\ldots\ldots\kappa_r!}\left(\frac{s_1}{1}\right)^{\kappa_1}\left(\frac{s_2}{2}\right)^{\kappa_2}\ldots\ldots\left(\frac{s_r}{r}\right)^{\kappa_r}.$$

Substituting in (61) by means of (60) and (58), we get

$$(62) \qquad \sum_{\kappa=0}^{\kappa=\infty}\frac{\kappa^r}{\kappa!}=e\sum\frac{r!}{\kappa_1!\kappa_2!\ldots\ldots\kappa_r!(1!)^{\kappa_1}(2!)^{\kappa_2}\ldots\ldots(r!)^{\kappa_r}}, \quad \kappa_1+2\kappa_2+\ldots\ldots+r\kappa_r=r.$$

We can also obtain a determinant form by using the determinant form for b_r,

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*Cf. The General Term of a Recurring Series, by Dr. L. E. Dickson, The American Mathematical Monthly, Vol. 10 (1903), page 223.

(42); (62) is the general solution.[*] A few particular results are given. If $r=1$, the right member of (62) becomes e, as it should, since $\kappa_1=1$.

If $r=2;\ \kappa_1+2\kappa_2=2,\ \kappa_1=2$	1	Sum
$\kappa_2=1$	1	$2e$
$r=3,\ \kappa_1+2\kappa_2+3\kappa_3=3,\ \kappa_1=3$	1	
$\kappa_1=1,\ \kappa_2=2$	3	
$\kappa_3=1$	1	$5e$
$r=4,\ \kappa_1+2\kappa_2+3\kappa_3+4\kappa_4=4,\ \kappa_1=4$	1	
$\kappa_1=2,\ \kappa_2=1$	6	
$\kappa_1=1,\ \kappa_3=1$	4	
$\kappa_2=2$	3	
$\kappa_4=1$	1	$15e$

1. THE MONOMIAL SYMMETRIC FUNCTIONS IN TERMS OF THE t'S.

Weight 1.

	t_1
(1)	1

Weight 2.

	t_1^2	t_2
(2)	−1	2
(1^2)	1	−1

Weight 3.

	t_1^3	t_1t_2	t_3
(3)	1	−3	3
(21)	−2	5	−3
(1^3)	1	−2	1

Weight 4.

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	−1	4	−2	−4	4
(31)	2	−7	2	7	−4
(2^2)	1	−4	3	2	−2
(21^2)	−3	10	−4	−7	4
(1^4)	1	−3	1	2	−1

Weight 5.

	t_1^5	$t_1^3t_2$	$t_1t_2^2$	$t_1^2t_3$	t_2t_3	t_1t_4	t_5
(5)	1	−5	15	5	−5	−5	5
(41)	−2	9	−7	−9	5	9	−5
(32)	−2	10	−11	−8	11	5	−5
(31^2)	3	−13	10	12	−8	−9	5
(2^21)	3	−14	14	10	−11	−7	5
(21^3)	−4	17	−14	−13	10	9	−5
(1^5)	1	−4	3	3	−2	−2	1

These tables are symmetrical with respect to the secondary diagonal. The sum of the numbers of the last column is unity, while that of the other columns is zero, because the sum of the monomial symmetric functions in the vertical column at the left of the table is, by definition, the complete symmetric function of weight w, t_w.

Multiplying the elements of the inverse table columnwise by those of the lines of the direct table (linewise), and so that the order is the same in both tables, and summing, we get zero, except when line and column belong to the same partition. Then the sum is unity. The reason for this is that the identity which so arises requires it. This property is true of all direct and inverse tables of the kind given in this paper. It is illustrated by the following arrangement of the direct and inverse tables for weight four.

[†] Cf. The American Mathematical Monthly, Vol. 9 (1902), pages 11 and 12. C. Smith's Treatise on Algebra, example 18, page 387. Loney's Trigonometry, Part II, page 9, example 7.

Direct Table (read vertically).

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	-1	4	-2	-4	4
(31)	2	-7	2	7	-4
(2^2)	1	-4	3	2	-2
(21^2)	-3	10	-4	-7	4
(1^4)	1	-3	1	2	-1

Inverse Table (read vertically).

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	1	1	1	1	1
(31)	4	3	2	2	1
(2^2)	6	4	3	2	1
(21^2)	12	7	4	3	1
(1^4)	24	12	6	4	1

Here the sum of the products of the elements of any two lines or of any two columns of the two tables is zero unless the two lines or the two columns are similarly placed, when the sum of the products of their elements is equal to unity. If two determinants be formed of the two arrays of the direct and inverse table, it thus appears that the product of these two determinants is unity. Hence the determinant of each array is also equal to unity numerically, when each determinant is integral; in other cases these determinants are reciprocals of each other.

2. THE t'S IN TERMS OF MONOMIAL SYMMETRIC FUNCTIONS. INVERSE TABLE.

Weight 1.

	(1)
t_1	1

Weight 2.

	(1^2)	(2)
t_2	1	1
t_1^2	2	1

Weight 3.

	(1^3)	(21)	(3)
t_3	1	1	1
t_1t_2	3	2	1
t_1^3	6	3	1

Weight 4.

	(1^4)	(21^2)	(2^2)	(31)	(4)
t_4	1	1	1	1	1
t_1t_3	4	3	2	2	1
t_2^2	6	4	3	2	1
$t_1^2t_2$	12	7	4	3	1
t_1^4	24	12	6	4	1

Weight 5.

	(1^5)	(21^3)	(2^21)	(31^2)	(32)	(41)	(5)
t_5	1	1	1	1	1	1	1
t_1t_4	5	4	3	3	2	2	1
t_3t_2	10	7	5	4	3	2	1
$t_3t_1^2$	20	13	8	7	4	3	1
$t_2^2t_1$	30	18	11	8	5	3	1
$t_1^3t_2$	60	33	18	13	7	4	1
t_1^5	120	60	30	20	10	5	1

In these tables the elements are symmetrical with respect to the secondary diagonal. The elements are all positive. The relations into which they enter with those of the direct table have already been mentioned.

3. THE t'S IN TERMS OF THE b'S.

Weight 1.

	b_1
t_1	-1

Weight 2.

	b_1^2	b_2
t_2	1	-1
t_1^2	1	

Weight 3.

	b_1^3	b_1b_2	b_3
t_3	-1	2	-1
t_1t_2	-1	1	
t_1^3	-1		

Weight 4.

	b_1^4	$b_1^2b_2$	b_2^2	b_1b_3	b_4
t_4	1	−3	1	2	−1
t_1t_3	1	−2	0	1	
t_2^2	1	−2	1		
$t_2t_1^2$	1	−1			
t_1^4	1				

Weight 5.

	b_1^5	$b_1^3b_2$	$b_1b_2^2$	$b_1^2b_3$	b_2b_3	b_1b_4	b_5
t_5	−1	4	−3	−3	2	2	−1
t_1t_4	−1	3	−1	−2	0	1	
t_2t_3	−1	3	−2	−1	1		
$t_1^2t_3$	−1	2	0	−1			
$t_1t_2^2$	−1	2	−1				
$t_1^3t_2$	−1	1					
t_1^5	−1						

In these tables the elements are contained within a right triangle. Symmetry is not present. Elements in the same column have the same sign. The sum of all the elements in a horizontal line (the last one excepted) vanishes, as it must according to (28). The sum of all the elements is $(-1)^w$ for the same reason. The relations to the inverse table are seen by writing the inverse as in 1. to read vertically.

4. THE *b*'S IN TERMS OF THE *t*'S. (INVERSE OF 3.)

In 3. interchange b and t, and we have the table, on account of (23) as worked out in (26) and (27).

5. THE *s*'S IN TERMS OF THE *t*'S. (THE *s*'S IN TERMS OF THE *b*'S.)

Weight 1.

	t_1
s_1	1

Weight 2.

	t_1^2	t_2
s_2	−1	2
s_1^2	1	

Weight 3.

	t_1^3	t_1t_2	t_3
s_3	1	−3	3
s_1s_2	−1	2	
s_1^3	1		

Weight 4.

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
s_4	−1	4	−2	−4	4
s_1s_3	1	−3	0	3	
s_2^2	1	−4	4		
$s_1^2s_2$	−1	2			
s_1^4	1				

Weight 5.

	t_1^5	$t_2t_1^3$	$t_2^2t_1$	$t_3t_1^2$	t_3t_2	t_1t_4	t_5
s_5	1	−5	5	5	−5	−5	5
s_4s_1	−1	4	−2	−4	0	4	
s_3s_2	−1	5	−6	−3	6		
$s_3s_1^2$	1	−3	0	3			
$s_2^2s_1$	1	−4	4				
$s_2s_1^3$	−1	2					
s_1^5	1						

Here the elements are contained in a right triangle. The sum of the elements in a horizontal line is 1, as it should be by (32) and (34).

If s is changed into $-s$, and t into b, these tables then express s's in terms of b's, on account of (35) and (36) as before observed. This amounts to changing t into b, and changing or not changing all the signs of the elements in a horizontal line according as the product of the s's at the left of the line is of odd or of even degree. And the sums of such collinear elements are accordingly ∓ 1.

6. THE t'S IN TERMS OF THE s'S. (THE b'S IN TERMS OF THE s'S.) (Inverse of 5.)

Weight 1.

	s_1
t_1	1

Weight 2.

	s_1^2	s_2
t_2	$\frac{1}{2}$	$\frac{1}{2}$
t_1^2	1	

Weight 3.

	s_1^3	$s_1 s_2$	s_3
t_3	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
$t_1 t_2$	$\frac{1}{2}$	$\frac{1}{2}$	
t_1^3	1		

Weight 4.

	s_1^4	$s_1^2 s_2$	s_2^2	$s_1 s_3$	s_4
t_4	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$
$t_1 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$	
t_2^2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		
$t_1^2 t_2$	$\frac{1}{2}$	$\frac{1}{2}$			
t_1^4	1				

Weight 5.

	s_1^5	$s_1^3 s_2$	$s_1 s_2^2$	$s_1^2 s_3$	$s_2 s_3$	$s_1 s_4$	s_5
t_5	$\frac{1}{120}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{5}$
$t_1 t_4$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	0	$\frac{1}{4}$	
$t_2 t_3$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$		
$t_1^2 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$			
$t_1 t_2^2$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$				
$t_1^3 t_2$	$\frac{1}{2}$	$\frac{1}{2}$					
t_1^5	1						

The sum of the horizontally collinear elements is 1, as it should be by (41).

If s is changed into $-s$, *i. e.*, if the signs of the elements of the columns are or are not changed according as the product of the s's heading the column is of odd or of even degree, and the t's are replaced by b's, we have then the table for the b's expressed in terms of the s's. In this case the sum of horizontally collinear elements, except for the last line, is zero, by (42) and (43).

TWO INFINITE SYSTEMS OF GROUPS GENERATED BY TWO OPERATORS OF ORDER FOUR.

By PROFESSOR G. A. MILLER. Stanford University.

If a, b, c represent respectively the orders of two operators and that of their product, the group generated by these operators is completely defined by the numbers a, b, c only in the following cases: (1) When one of these three numbers is unity; (2) When two of them are equal to 2 and the third greater than 1; (3) When they have one of the following three sets of values: (2, 3, 3), (2, 3, 4), (2, 3, 5). The corresponding groups are respectively cyclic, dihedral rotation, tetrahedral, octahedral, and icosahedral.*

The abstract definitions and the laws of combination of these groups are so simple that they furnish very interesting examples in the logic of algebra. We proceed to consider two other very elementary systems of groups. Let s_1, s_2, represent two non-commutative operators of order four such that

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*Burnside, Theory of groups of finite order, 1897, p. 291; Miller, *Bulletin of American Mathematical Society*, Vol. 7, 1901, p. 424; *American Journal of Mathematics*, Vol. 24, 1902, p. 96.